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A relation between one-point and multi-point Seshadri constants

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Abstract

T. Szemberg proposed in 2001 a generalization to arbitrary varieties of M. Nagata's 1959 open conjecture, which claims that the Seshadri constant of $r \geqslant 9$ very general points of the projective plane is maximal. Here we prove that Nagata's original conjecture implies Szemberg's for all smooth surfaces X with an ample divisor L generating $NS(X)$ and such that L^2 is a square.

More generally, we prove the inequality

$$\varepsilon_{n-1}(L, r) \geqslant \varepsilon_{n-1}(L, 1)\varepsilon_{n-1}\big(\mathcal{O}_{\mathbb{P}^n}(1), r\big),$$

where $\varepsilon_{n-1}(L, r)$ stands for the $(n-1)$-dimensional Seshadri constant of the ample divisor L at r very general points of a normal projective variety X, and $n = \dim X$.

Keywords: Seshadri constant; Nagata's conjecture

1. Introduction

Let X be a normal projective variety of dimension n over an algebraically closed field k, and L an ample divisor. Given r points $p_1, \dots, p_r \in X$ and an integer $1 \leqslant d \leqslant n$, the d-dimensional Seshadri constant of L at the points $p_1, \dots, p_r$ is the real number

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$$\varepsilon_d(L, p_1, \dots, p_r) = \sqrt[d]{\inf_Z \left\{ \frac{L^d \cdot Z}{\sum \operatorname{mult}_{p_i} Z} \right\}},$$

where Z runs over all positive d-dimensional cycle s (see [1, 1.1], or [2] for the original definition). As L is ample, we have $L^d \cdot Z > 0$ for all Z, so $(L^d \cdot Z)/(\sum \operatorname{mult}_{p_i} Z) \in \mathbb{R}_+ \cup \infty$. Moreover, there exist Z which contain some point p_i, and therefore the Seshadri constant is indeed a finite real number.

Most work on Seshadri constants deals with the $d = 1$ case, and usually one writes $\varepsilon(L, p_1, \dots, p_r) = \varepsilon_1(L, p_1, \dots, p_r)$; we shall be concerned here with the codimension 1 case ($d = n - 1$). Also, we use the shorthand notation $\varepsilon_d(L, r) = \varepsilon_d(L, p_1, \dots, p_r)$ for very general points $p_1, \dots, p_r$ (i.e., in the intersection of countably many Zariski open subsets of X^r) which is the case we are mostly interested in.

In connection with his solution to the fourteenth problem of Hilbert, Nagata posed in [3] (in different terminology) the following conjecture concerning Seshadri constants of the plane:

Conjecture 1 (Nagata). *If $r \geqslant 9$, then*

$$\varepsilon\big(\mathcal{O}_{P^2}(1), r\big) = 1/\sqrt{r}.$$

If $r = s^2$ is a square, then it is not hard to prove that the conjecture is true, and in fact the point of Nagata's stronger result for the $r = s^2 > 9$ case is that the infimum appearing in the definition of the Seshadri constant is not achieved by any plane curve. In a variety of dimension n it is also not difficult to prove that

$$\varepsilon(L, p_1, \dots, p_r) \leqslant \sqrt[n]{\frac{L^n}{r}}$$

for every set of r points (see [4, Remark 1], for example), so Nagata's conjecture claims that the Seshadri constant of a very general set of $r \geqslant 9$ points in the plane is maximal. All available information on Seshadri constants (see [5–11] for the case of surfaces, [12–14] for dimension $n > 2$) suggests that, in fact, in an arbitrary variety, for r large enough, the Seshadri constant of r very general points is maximal. This led Szemberg to propose in [1] the following generalization:

Conjecture 2 (Nagata–Szemberg). *Given a smooth variety of dimension n and L an ample divisor on X, there exists a number $r_0 = r_0(X, L)$ such that for every $r \geqslant r_0$*

$$\varepsilon(L, r) = \sqrt[n]{\frac{L^n}{r}}.$$

This note is devoted to the following result, that gives a lower bound for $(n-1)$-dimensional Seshadri constants of r very general points in a variety, relating them to the analogous constants of one point in the same variety and of r points in projective space:

Theorem 3. *Let X be a normal projective variety of dimension $n \geqslant 2$, L an ample divisor. Then for every smooth point $p \in X$ and $r \geqslant 1$,*

$$\varepsilon_{n-1}(L, r) \geqslant \varepsilon_{n-1}(L, p)\varepsilon_{n-1}\big(\mathcal{O}_{\mathbb{P}^n}(1), r\big).$$

Combining it with known results on the value of the Seshadri constants in projective space, Theorem 3 implies more explicit relations between r-point and 1-point Seshadri constants. The consequences support the Nagata–Szemberg conjecture, especially in the case of surfaces (note that for surfaces, $(n-1)$-dimensional Seshadri constants are the usual Seshadri constants).

Corollary 4. *Suppose $r = s^n$ for some integer s. Then for every smooth point $p \in X$ and $r \geqslant 1$,*

$$\varepsilon_{n-1}(L, r) \geqslant \varepsilon_{n-1}(L, p)/s.$$

Proof. Use G.V. Choodnovsky's result [13] that $\varepsilon_{n-1}(\mathcal{O}_{\mathbb{P}^n}(1), s^n) = 1/s$. □

If r is not the nth power of an integer, then we do not know the exact value of $\varepsilon_{n-1}(\mathcal{O}_{\mathbb{P}^n}(1), r)$ (there is a conjecture similar to Nagata's, posed by Choodnovsky in the same paper [13], and by A. Iarrobino in [15]). However, B. Harbourne pointed out that, using results of J. Alexander and A. Hirschowitz and of M. Hochster and C. Huneke, one can prove an asymptotically optimal bound for $\varepsilon_{n-1}(\mathcal{O}_{\mathbb{P}^n}(1), r)$. Combining it with Theorem 3, in Section 2 we prove the following asymptotic bound for $\varepsilon_{n-1}(L, r)$ that depends only on the $(n-1)$-dimensional Seshadri constant of L at a smooth point $p \in X$:

Corollary 5. *For every $\varepsilon > 0$ there exists an integer s, depending only on ε and n, such that for every smooth point $p \in X$ and $r \geqslant s$,*

$$\varepsilon_{n-1}(L, r)\sqrt[n]{r} \geqslant \varepsilon_{n-1}(L, p) - \varepsilon.$$

Corollary 6. *If X is a normal projective surface, L an ample divisor on X, then for every smooth $p \in X$ and $r \geqslant 9$, Nagata's conjecture implies that*

$$\varepsilon(L, r) \geqslant \frac{\varepsilon(L, p)}{\sqrt{r}}.$$

Observe that this tells us that Nagata's conjecture (on the plane) implies the Nagata–Szemberg conjecture on a large family of surfaces. Indeed, the obtained bound is equal to $\varepsilon(L, p)/\sqrt{L^2}$ times the conjectured value of $\varepsilon(L, r)$, so we have the following:

Corollary 7. *If X is a normal projective surface, L an ample divisor on X and $p \in X$ is a smooth point such that $\varepsilon(L, p) = \sqrt{L^2}$, then Nagata's conjecture implies the Nagata–Szemberg conjecture on (X, L) with $r_0(X, L) \leqslant 9$.*

In particular, we can apply this to complex surfaces with Picard number 1, using A. Steffens' result [4, Proposition 1], which says that if L is an ample generator of $NS(X)$ then $\varepsilon(L, p) \geqslant \lfloor\sqrt{L^2}\rfloor$ for very general points:

Corollary 8. *Let X be a smooth projective surface defined over $\mathbb{C}$, L an ample generator of $NS(X)$ and assume $L^2 = d^2$ is a square. Then Nagata's conjecture implies the Nagata–Szemberg conjecture on X with $r_0(X, L) \leqslant 9$.*

In particular, the Seshadri constant of $r \geqslant 9$ very general points on a complex surface X with Picard number equal to 1, is maximal if both L^2 and the number of points r are squares. This can be compared to B. Harbourne's result [11, I.1] (over an arbitrary base field and with no assumption on the Picard number) that the Seshadri constant is maximal whenever L is very ample, rL^2 is a square, and $r \geqslant L^2$.

Also, known bounds approximating Nagata's conjecture give new bounds on surfaces; for instance, H. Tutaj-Gasińska's bound in [16] showing that $\varepsilon(\mathcal{O}_{\mathbb{P}^2_{\mathbb{C}}}(1), r) \geqslant 1/(\sqrt{r + 1/12})$ gives the following:

Corollary 9. *If X is a normal projective surface defined over $\mathbb{C}$, then for every smooth point $p \in X$ and $r > 9$,*

$$\varepsilon(L, r) \geqslant \frac{\varepsilon(L, p)}{\sqrt{r + \frac{1}{12}}}.$$

In a similar vein, Harbourne's bounds on Seshadri constants of $\mathbb{P}^2$ in [17] and [11] imply that

Corollary 10. *Let X be a normal projective surface, $p \in X$ a smooth point and $r \geqslant 1$. Then for every pair of integers $1 \leqslant s \leqslant r$, $1 \leqslant d$, it holds*

$$\varepsilon(L, r) \geqslant \begin{cases} \frac{s}{rd}\varepsilon(L, p), & \text{if } s^2 \leqslant rd^2, \\ \frac{d}{s}\varepsilon(L, p), & \text{if } s^2 \geqslant rd^2. \end{cases}$$

It should be mentioned, however, that both Corollaries 9 and 10 are usually weaker (but stronger for some surfaces and numbers of points) than Harbourne's results on algebraic surfaces of [11], where he gets the bounds

$$\varepsilon(L, r) \geqslant \begin{cases} \frac{s}{rd}, & \text{if } s^2 \leqslant rd^2L^2, \\ \frac{dL^2}{s}, & \text{if } s^2 \geqslant rd^2L^2 \end{cases}$$

for very ample L, assuming moreover that $r \geqslant L^2$.

The proof of Theorem 3 is based on the idea, due to L. Évain (see [18]) that r-point Seshadri constants of the plane can be computed by means of homothetic collisions of fat points. For convenience of the exposition we express this, generalized to n-dimensional projective space, in terms of the order of a nonreduced curve singularity, rather than collisions. Then, we observe that it is enough to know the formal germ of such a singularity, and the fact that the completion of the local ring at a smooth point of a variety is a ring of formal power series that only depends on the dimension of the variety, to obtain the bound.

It might be interesting to note that the Viro method developed by E. Shustin in [19] can also be used to relate the existence of singular curves on $\mathbb{P}^2$ with the existence of singular curves in algebraic surfaces, see for instance [20, §5] or [8, 3.A].

2. Singularities of arrangements of multiple lines

To lighten notations for rings of polynomials and of power series, we write $\mathbf{x} = (x_0, \dots, x_n)$ for a collection of variables, so $k[\mathbf{x}]$ and $k[\![\mathbf{x}]\!]$ denote $k[x_0, \dots, x_n]$ and $k[\![x_0, \dots, x_n]\!]$, respectively. Also, $\mathfrak{m}$ and $\hat{\mathfrak{m}}$ will be the maximal ideals generated by $x_0, \dots, x_n$ in $k[\mathbf{x}]$ and $k[\![\mathbf{x}]\!]$, respectively, and for every point $p = [\xi_0 : \dots : \xi_n]$ in projective n-space, we use the notation I_p (respectively $\hat{I}_p$) for the ideal generated by the 2×2 minors of the matrix

$$\begin{pmatrix} \xi_0 & \dots & \xi_n \\ x_0 & \dots & x_n \end{pmatrix}$$

in $k[\mathbf{x}]$ (respectively in $k[\![\mathbf{x}]\!]$).

Given distinct points $p_1, \dots, p_r \in \mathbb{P}^n$ and $\mathbf{m} = (m_1, \dots, m_r) \in \mathbb{Z}^r_{\geqslant 0}$, we define $\alpha_{\mathbf{m}}(p_1, \dots, p_r)$ to be the minimal degree of a homogeneous polynomial vanishing to order m_i at p_i. As I_p is homogeneous for all p, this number coincides with the maximal integer α such that

$$I = \bigcap_{i=1}^{r} I_{p_i}^{m_i} \subset \mathfrak{m}^\alpha,$$

or equivalently, such that $\hat{I} = \bigcap \hat{I}_{p_i}^{m_i} \subset \hat{\mathfrak{m}}^\alpha$. In other words, $\alpha_{\mathbf{m}}(p_1, \dots, p_r)$ is the order at the origin of $\mathbb{A}^{n+1}$ of the arrangement of multiple lines defined by I (which is the affine cone over the fat point scheme consisting of the points p_i with multiplicities m_i).

Remark 11. The definition of α and the $(n-1)$-dimensional Seshadri constants immediately give that $\forall \mathbf{m}$,

$$\alpha_{\mathbf{m}}(p_1, \dots, p_r) \geqslant \bigl(\varepsilon_{n-1}\bigl(\mathcal{O}_{\mathbb{P}^n}(1), p_1, \dots, p_r\bigr)\bigr)^{n-1} \sum_{i=1}^{r} m_i.$$

Let X be a variety of dimension $n \geqslant 2$, $q \in X$ a smooth point, and fix uniformizing parameters $x_1, \dots, x_n$ in some neighborhood V centered at q. So (see [21, §III.6], for example) the germs of $x_1, \dots, x_n$ at q generate the maximal ideal in the local ring $\mathcal{O}_{X,q}$ and the morphism of k-algebras $k[\mathbf{x}] \hookrightarrow \mathcal{O}_X(V)$ determines an étale morphism $\varphi : V \to \mathbb{A}^n \cong T_q X$ (where $V \subset X$ is open) and an isomorphism $k[\![\mathbf{x}]\!] \to \widehat{\mathcal{O}}_{X,q}$.

To every $p = [\xi_0 : \cdots : \xi_n] \in \mathbb{P}^n$ such that $\xi_0 \neq 0$ and $p \neq [1 : 0 : \cdots : 0]$ (so that $p' = [\xi_1 : \cdots : \xi_n] \in \mathbb{P}^{n-1}$), we shall assign an irreducible curve C_p, smooth at q, and a regular parameter $\bar{x}_p \in \mathcal{O}_{C_p,q}$. Define C_p as the closure of the component through q of $\varphi^{-1}(L_{p'})$, where $L_{p'} \subset \mathbb{A}^n$ is the affine cone over p', i.e., the line through the origin in the direction determined by p'. As φ is étale, C_p is smooth at q, and the ideal $I_{C_p} \subset \mathcal{O}_{X,q}$ of its germ at q is generated by the 2×2 minors of the matrix

$$\begin{pmatrix} \xi_1 & \dots & \xi_n \\ x_1 & \dots & x_n \end{pmatrix}.$$

Now consider $x_p = x_i \xi_0 / \xi_i$ for some $\xi_i \neq 0$, $i \geqslant 1$. It is easy to see that the restriction $\bar{x}_p$ of x_p to C_p does not depend on the choice of i, and that it is a uniformizing parameter for the curve. Also, abusing notation, in $\mathcal{O}_{C_p \times X,(q,q)} = \mathcal{O}_{C_p,q} \otimes \mathcal{O}_{X,q}$ we write $x_p = \bar{x}_p \otimes 1$, $x_1 = 1 \otimes x_1, \dots,$ and $x_n = 1 \otimes x_n$; then the ideal of the germ of the diagonal $\Delta(C_p) \subset C_p \times X$ is generated by the 2×2 minors of the matrix

$$\begin{pmatrix} \xi_0 & \xi_1 & \dots & \xi_n \\ x_p & x_1 & \dots & x_n \end{pmatrix},$$

in other words, $\Delta(C_p)$ is the closure of the component through the origin of $\psi^{-1}(L_p)$, where L_p is the affine cone over $p \in \mathbb{P}^n$, i.e., a line $L_p \subset \mathbb{A}^{n+1} = T_{(q,q)}(C_p \times X)$, and ψ is the étale morphism given by the parameters $x_p, x_1, \dots, x_n$.

With these notations, Theorem 3 follows from the more precise proposition:

Proposition 12. *Let X be a variety of dimension $n \geqslant 2$, L an ample divisor, $q \in X$ a smooth point, $p_1, \dots, p_r \in \mathbb{P}^n \setminus [1 : 0 : \cdots : 0]$ distinct points not on the hyperplane $\xi_0 = 0$. Then for very general points $q_k \in C_{p_k}$ we have*

$$\varepsilon_{n-1}(L, q_1, \dots, q_r) \geqslant \varepsilon_{n-1}(L, q)\varepsilon_{n-1}\big(\mathcal{O}_{\mathbb{P}^n}(1), p_1, \dots, p_r\big).$$

Proof. First note that due to the semicontinuity of multiplicity (see [22, §8] or [23, §3]), for each component $\mathcal{H}$ of the Hilbert scheme of hypersurfaces in X, and each system of multiplicities $\mathbf{m}$, the sets of points $(q_1, \dots, q_r)$ such that there is $Y \in \mathcal{H}$ with multiplicity $\geqslant m_i$ at q_i form a Zariski-closed subset of X^r. Thus, it will be enough to prove that, given $\mathcal{H}$ and $\mathbf{m}$, the existence of $Y \in \mathcal{H}$ with multiplicity $\geqslant m_k$ at q_k for general $q_k \in C_{p_k}$ implies

$$Y \cdot L^{n-1} \geqslant \big(\varepsilon_{n-1}(L, q)\varepsilon_{n-1}\big(\mathcal{O}_{\mathbb{P}^n}(1), p_1, \dots, p_r\big)\big)^{n-1} \sum_{k=1}^{r} m_k.$$

So, fix $\mathcal{H}$ and $\mathbf{m}$, and assume that for general points $q_k \in C_{p_k}$ there is a hypersurface $Y \in \mathcal{H}$ going through q_k with multiplicity at least m_k.

In the local ring of $\prod C_{p_k}$ at $\Delta(q) = (q, \dots, q)$, the $1 \otimes \cdots \otimes \bar{x}_{p_k} \otimes \cdots \otimes 1$, $k = 1, \dots, r$, form a regular system of parameters; abusing notation we call them simply x_{p_k}. Let $\Gamma \subset \prod C_{p_k}$ be the irreducible curve defined locally by the equations $x_{p_1} = x_{p_i}$, $i = 2, \dots, r$, which is obviously smooth at $\Delta(q)$, and admits $x_0 := \bar{x}_{p_1} \in \mathcal{O}_{\Gamma,\Delta(q)}$ as a local parameter.

For every $p_k = [\xi_{k,0} : \cdots : \xi_{k,n}]$, the ideal $J_{p_k} \subset \mathcal{O}_{\Gamma \times X, \Delta(q)}$ generated by the 2×2 minors of the matrix

$$\begin{pmatrix} \xi_{k,0} & \dots & \xi_{k,n} \\ x_0 & \dots & x_n \end{pmatrix},$$

defines the germ of a curve C'_{p_k} (the preimage of a line in $\mathbb{A}^{n+1} \cong T_{\Delta(q)}(\Gamma \times X)$) whose projection to X is exactly C_{p_k}; more precisely, the fiber of C'_{p_k} over $\gamma = (q_1, \dots, q_n) \in \Gamma$ is $q_k \in C_{p_k}$.

Consider now the diagonals $\Delta_{i,j} = \{(q_1, \dots, q_r) \in \prod C_{p_k} \mid q_i = q_j\}$, $\Delta = \bigcup \Delta_{i,j}$, and $U = \Gamma \setminus \Delta$. As the points $p_1, \dots, p_r$ are distinct, U is a nonempty open subset of Γ. The assumption on the existence of Y tells us that there is an effective Weil divisor $\mathbf{Y}_U \subset U \times X$ flat over U whose fiber over $(q_1, \dots, q_r) \in U$ belongs to $\mathcal{H}$ and has multiplicity at least m_i at q_i. By the smoothness of Γ at $\Delta(q)$, $\mathbf{Y}_U$ can be extended to a flat family $\mathbf{Y} \subset (U \cup \{\Delta(q)\}) \times X$, and then the condition on the multiplicity of the fibers of $\mathbf{Y}$ means that $\mathbf{Y}$ contains the arrangement of multiple curves whose germ at $\Delta(q)$ is defined by the ideal

$$J = \bigcap_{k=1}^{r} J_{p_k}^{m_k},$$

and this implies that the fiber Y_q of $\mathbf{Y}$ over $\Delta(q)$ has multiplicity at least equal to the order of this arrangement at $\Delta(q)$, i.e., at least equal to the maximal integer α such that $J \subset (x_0, \dots, x_n)^\alpha$. This can be computed equivalently as the order of the completion $\hat{J}$ in $\widehat{\mathcal{O}}_{\mathbb{A}^1 \times X, (0,q)} \cong k[\![\mathbf{x}]\!]$ but, by construction of the curves C'_{p_k}, one has $\hat{J} = \hat{I}$ as defined above, so this order is exactly $\alpha_{\mathbf{m}}(p_1, \dots, p_r)$.

We remark also that, by the smoothness of X in a neighborhood of q, $\mathbf{Y}$ is defined by a principal ideal at q, so we get a Weil divisor $Y'_q \subseteq Y_q$ with

$$\begin{aligned} L^{n-1} \cdot Y'_q &= L^{n-1} \cdot Y, \\ \operatorname{mult}_q Y'_q &\geqslant \alpha_{\mathbf{m}}(p_1, \dots, p_r), \end{aligned}$$

which together with the definition of the 1-point Seshadri constant, gives $L^{n-1} \cdot Y \geqslant \varepsilon_{n-1}(X, q)\alpha_{\mathbf{m}}(p_1, \dots, p_r)$, and then it is enough to apply the bound of Remark 11. □

Proof of Corollary 5. By Theorem 3 it is enough to see that

$$\lim_{r \to \infty} \varepsilon_{n-1}\big(\mathcal{O}_{\mathbb{P}^n}(1), r\big)\sqrt[n]{r} \geqslant 1$$

(which in fact means that one has an equality, the converse inequality being well-known). More precisely, we shall prove that given $k > 0$ there exists $s_k = s_k(n)$ such that if $r \geqslant s_k$ then for all $\mathbf{m} = (m_1, \dots, m_r)$ and general points $p_1, \dots, p_r$,

$$\alpha_{\mathbf{m}}(p_1, \dots, p_r) \geqslant \frac{\sum m_i}{\sqrt[n]{r^{n-1}}} \cdot \frac{k+1}{k+n}.$$

So let F be a homogeneous polynomial defining a hypersurface of degree d in $\mathbb{P}^n$ which has multiplicity m_i at p_i for general points $p_1, \dots, p_r$. Then, by the genericity of the points, for every permutation $\sigma \in \mathcal{S}_r$ there is a polynomial F_σ which has multiplicity m_i at the point $p_{\sigma(i)}$. Therefore $G = \prod_{\sigma \in \mathcal{S}_r} F_\sigma$ is a polynomial of degree $D = r!\,d$ which has (the same) multiplicity $M = (r-1)! \sum m_i$ at $p_1, \dots, p_r$, and G^{k+n} has degree $(k+n)D$ and multiplicity $(k+n)M$ at each point. By [24, Theorem 1.1(a)], applied to the ideal I of the (reduced) scheme $\{p_1, \dots, p_r\}$, this implies the existence of hypersurfaces of degree $t \leqslant (k+n)D/M$ with multiplicity at least $k+1$ at $p_1, \dots, p_r$. Now write $\mathbf{m}' = (k+1, \dots, k+1)$; by [12, Corollary 1.2], there is $s_k(n)$ such that if $r \geqslant s_k(n)$ then $\alpha_{\mathbf{m}'}(p_1, \dots, p_r) \geqslant (k+1)\sqrt[n]{r}$ (again, because the points are general). Therefore we get $(k+n)D/M \geqslant (k+1)\sqrt[n]{r}$ and

$$\frac{d}{\sum m_i} = \frac{D}{rM} \geqslant \frac{1}{\sqrt[n]{r^{n-1}}} \cdot \frac{k+1}{k+n},$$

as desired. □

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